

DOMINATING SURFACE GROUP REPRESENTATIONS AND DEFORMING CLOSED ANTI-DE SITTER 3-MANIFOLDS

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ABSTRACT. Let S be a closed oriented surface of negative Euler characteristic and M a complete contractible Riemannian manifold. A Fuchsian representation $j : \pi_1(S) \rightarrow \text{Isom}^+(\mathbb{H}^2)$ *strictly dominates* a representation $\rho : \pi_1(S) \rightarrow \text{Isom}(M)$ if there exists a (j, ρ) -equivariant map from \mathbb{H}^2 to M that is λ -Lipschitz for some $\lambda < 1$. In a previous paper by Deroin–Tholozan, the authors construct a map Ψ_ρ from the Teichmüller space $\mathcal{T}(S)$ of the surface S to itself and prove that, when M has sectional curvature ≤ -1 , the image of Ψ_ρ lies (almost always) in the domain $D(\rho)$ of Fuchsian representations strictly dominating ρ . Here we prove that $\Psi_\rho : \mathcal{T}(S) \rightarrow D(\rho)$ is a homeomorphism. As a consequence, we are able to describe the topology of the space of pairs of representations (j, ρ) from $\pi_1(S)$ to $\text{Isom}^+(\mathbb{H}^2)$ with j Fuchsian strictly dominating ρ . In particular, we obtain that its connected components are classified by the Euler class of ρ . The link with anti-de Sitter geometry comes from a theorem of Kassel stating that those pairs parametrize deformation spaces of anti-de Sitter structures on closed 3-manifolds.

INTRODUCTION

0.1. Closed AdS 3-manifolds. An *anti-de Sitter* (AdS) manifold is a smooth manifold equipped with a Lorentz metric of constant negative sectional curvature. In dimension 3, those manifolds are locally modelled on $\text{PSL}(2, \mathbb{R})$ with its Killing metric, whose isometry group identifies to a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ extension of $\text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ acting by left and right translations, i.e.

$$(g_1, g_2) \cdot x = g_1 x g_2^{-1}$$

for all $(g_1, g_2) \in \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$ and all $x \in \text{PSL}(2, \mathbb{R})$.

The specific interest for anti-de Sitter structures on 3-manifolds takes its roots in the geometrization program of Thurston. Following Thurston’s classification of the eight Riemannian geometries in dimension 3, Scott [23] proved that a large class of closed 3-manifolds, namely, most of Siefert fiber spaces, could be modelled on the sixth geometry, that is, $\widetilde{\text{PSL}(2, \mathbb{R})}$ with a left invariant Riemannian metric that is also $\widetilde{PSO(2)}$ -invariant on the right. In particular, those manifolds carry an AdS structure, called standard. At first, it was conjectured that all closed AdS 3-manifolds had to be standard (the conjecture still holds in higher dimension, see [30]). However, it turned out progressively that the space of AdS structures on closed 3-manifolds is much richer than the space of standard structures.

First, Goldman [8] noticed that, in some cases, a standard structure could be deformed into a non-standard one. Namely, he proved that, if S is a closed surface of negative Euler characteristic, $j : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{R})$ a discrete and faithful representation and $\rho : \pi_1(S) \rightarrow$

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$\mathrm{PSL}(2, \mathbb{R})$ a representation that is sufficiently close to the trivial representation, then the embedding

$$(j, \rho)(\pi_1(S)) = \{(j(\gamma), \rho(\gamma)), \gamma \in \pi_1(S)\} \subset \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$$

acts properly discontinuously and cocompactly on $\mathrm{PSL}(2, \mathbb{R})$. The quotient is nonstandard whenever the image of ρ is not included in a compact subgroup.

Recall that a Lorentz metric on a manifold possesses a geodesic flow, and the metric is called *complete* if its geodesics run for all time. In dimension 3, a complete AdS manifold is a quotient of $\mathrm{PSL}(2, \mathbb{R})$ by a subgroup of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ acting properly discontinuously on $\mathrm{PSL}(2, \mathbb{R})$. In [17], Kulkarni and Raymond addressed the question of describing all complete AdS 3-manifolds. Their first assertion is that complete compact AdS 3-manifolds have *finite level*, that is, they are actually quotients of a finite cover of $\mathrm{PSL}(2, \mathbb{R})$. (A correct proof in every dimension was given by Zeghib [30].) Therefore, closed AdS 3-manifolds are cyclic coverings of a quotient of $\mathrm{PSL}(2, \mathbb{R})$ by the action of a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$. Kulkarni and Raymond then proved that such quotients have, up to a finite cover, the form of Goldman's quotients:

Theorem (Kulkarni–Raymond, [17]). *Let Γ be a discrete group acting properly discontinuously and cocompactly on $\mathrm{PSL}(2, \mathbb{R})$ via a faithful representation $h : \Gamma \rightarrow \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$. Assume moreover that Γ is torsion-free. Then Γ is the fundamental group of a closed surface of negative Euler characteristic and*

$$h = (j, \rho),$$

with either j or ρ discrete and faithful.

Remark 0.1. This theorem was generalized by Kobayashi [16] and Kassel [13] to quotients of any rank 1 semisimple Lie group G under the action of some subgroup of $G \times G$ by left and right translations.

Remark 0.2. Contrary to the Riemannian setting, a Lorentz metric on a closed manifold may not be complete. However, Klingler proved later [15], generalizing Carrière's theorem [3], that closed Lorentz manifolds of constant curvature are complete. Hence the completeness assumption in Kulkarni–Raymond's work can be removed.

Recall that, as an application of Selberg's lemma [24], if there exists a faithful representation of Γ into $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$, then Γ admits a torsion-free finite index subgroup. Therefore, Kulkarni–Raymond's theorem describes all closed AdS 3-manifolds up to a finite cover. A consequence of their theorem is that any closed AdS 3-manifold is finitely covered by a circle bundle over a surface, and is itself a Siefert fiber space.

We now fix a closed oriented surface S of negative Euler characteristic $\chi(S)$. A given pair (j, ρ) of representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R})$ does not necessarily induce a proper action of $\pi_1(S)$ on $\mathrm{PSL}(2, \mathbb{R})$. For instance, if $\rho(\gamma)$ is conjugate to $j(\gamma)$ for some $\gamma \neq 1$, then the infinite subgroup generated by γ acts via (j, ρ) with a fixed point. A pair (j, ρ) is called *admissible* if $(j, \rho)(\pi_1(S))$ acts properly discontinuously and cocompactly on $\mathrm{PSL}(2, \mathbb{R})$.

Let (j, ρ) be an admissible pair. One easily checks that (ρ, j) is also admissible. If σ denotes the outer automorphism of $\mathrm{PSL}(2, \mathbb{R})$ given by

$$\sigma(g) = {}^t g^{-1},$$

then $(\sigma \circ j, \sigma \circ \rho)$ is also admissible. Finally, for any $g_1, g_2 \in \mathrm{PSL}(2, \mathbb{R})$, the pair $(\mathrm{Ad}_{g_1} \circ j, \mathrm{Ad}_{g_2} \circ \rho)$ is still admissible. (Ad_{g_i} denotes the conjugation by g_i on $\mathrm{PSL}(2, \mathbb{R})$.) We will denote by $\mathrm{Adm}(S)$ the quotient of the space of admissible pairs of representations of $\pi_1(S)$ by these transformations. The main objective of this article is to describe the topology of $\mathrm{Adm}(S)$ and, in particular, classify its connected components. This will answer one of the many interesting questions of the recent survey: *Some open questions in anti-de Sitter geometry* ([2], section 2.3).

Recall that, by the work of Goldman [9], connected components of the space of representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R})$ are classified by an integer called the *Euler class*. It can take any value between $\chi(S)$ and $-\chi(S)$ and is extremal (i.e. equal to $\pm\chi(S)$) if and only if the representation is *Fuchsian*, that is, discrete and faithful. The postcomposition with the outer automorphism σ permutes the components of extremal Euler class. Therefore, up to switching the factors and postcomposing with σ , we can assume that any admissible pair (j, ρ) has j Fuchsian of Euler class $\chi(S)$. From now on, we will consider $\mathrm{Adm}(S)$ as the space of admissible pairs (j, ρ) with j of Euler class $\chi(S)$ modulo the action of $\mathrm{PSL}(2, \mathbb{R}) \times \mathrm{PSL}(2, \mathbb{R})$ by conjugation.

If j is Fuchsian and ρ takes values into a compact subgroup of $\mathrm{PSL}(2, \mathbb{R})$, then the pair (j, ρ) is clearly admissible, and the quotient is standard. According to [8], there are some non-standard admissible pairs with ρ in the connected component of the trivial representation. Kulkarni and Raymond believed that all non-standard structures could be obtained by deforming standard ones. It was proven wrong by Salein.

Denote by \mathbb{H}^2 the Poincaré half-plane, and by g_P the Poincaré metric on \mathbb{H}^2 , of curvature -1 . The group $\mathrm{PSL}(2, \mathbb{R})$ acts on \mathbb{H}^2 by homographies and identifies with the group of orientation preserving isometries of (\mathbb{H}^2, g_P) . Salein [20] noticed that a sufficient condition for a pair (j, ρ) to be admissible is that j *strictly dominates* ρ , in the sense that there exists a (j, ρ) -equivariant map from (\mathbb{H}^2, g_P) to (\mathbb{H}^2, g_P) which is λ -lipschitz for some $\lambda < 1$. As a consequence, he constructed admissible pairs (j, ρ) with ρ of any non-extremal Euler class. In particular, those admissible pairs cannot be continuously deformed into standard ones. Lastly, Kassel proved in [14] that Salein's sufficient condition for admissibility is also necessary:

Theorem (Kassel). *Let S be a closed surface of negative Euler characteristic and j, ρ two representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{R})$, with j Fuchsian. Then the pair (j, ρ) is admissible if and only if there exists a (j, ρ) -equivariant map from (\mathbb{H}^2, g_P) to (\mathbb{H}^2, g_P) that is λ -Lipschitz for some $\lambda < 1$.*

0.2. Dominated representations. Kassel's criterion for admissibility raises many questions that may turn out to be interesting beyond the scope of 3-dimensional AdS geometry. Indeed, one can extend the definition of domination to a more general setting. Consider a closed surface S of negative Euler characteristic, and a contractible Riemannian manifold (M, g_M) . Let $j : \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian representation, and ρ a representation of $\pi_1(S)$ into $\mathrm{Isom}(M, g_M)$. Since M is contractible, there always exists a smooth (j, ρ) -equivariant map from \mathbb{H}^2 to M . Since S is compact, this map is Lipschitz. We can thus define the *minimal Lipschitz constant*

$$\mathrm{Lip}(j, \rho) = \inf \{ \lambda \in \mathbb{R}_+ \mid \exists f : \mathbb{H}^2 \rightarrow M \text{ } (j, \rho)\text{-equivariant and } \lambda\text{-lipschitz} \}.$$

We then say that j strictly dominates ρ if $\mathrm{Lip}(j, \rho) < 1$.

0.2.1. Length spectrum. A related notion of domination comes from the comparison of the length spectrum of j and ρ . Recall that if g is an isometry of a metric space (M, d) , the *translation length* of g is the number

$$l(g) := \inf_{x \in M} d(x, g \cdot x).$$

The length spectrum of a representation $\rho : \pi_1(S) \rightarrow \text{Isom}(M)$ is the function that associates to $\gamma \in \pi_1(S)$ the translation length of $\rho(\gamma)$. We say that the length spectrum of j strictly dominates the length spectrum of ρ if there exists a constant $\lambda < 1$ such that for all $\gamma \in \pi_1(S)$,

$$l(\rho(\gamma)) \leq \lambda l(j(\gamma)).$$

It is straightforward that if a Fuchsian representation j strictly dominates ρ , then the length spectrum of j strictly dominates the length spectrum of ρ . The converse is highly non-trivial, but happens to be true in the cases we are going to consider, by a theorem of Guéritaud–Kassel:

Theorem (Guéritaud–Kassel, [10]). *Let S be a closed oriented surface of negative Euler characteristic, j a Fuchsian representation of $\pi_1(S)$ into $\text{PSL}(2, \mathbb{R})$, and ρ a representation of $\pi_1(S)$ into the isometry group of some smooth complete simply connected Riemannian manifold of sectional curvature ≤ -1 . Then j strictly dominates ρ if and only if the length spectrum of j strictly dominates the length spectrum of ρ .*

Remark 0.3. In [10], Guéritaud and Kassel study the minimal Lipschitz constant $\text{Lip}(j, \rho)$ and its relation with the comparison of the length spectra of j and ρ in a much broader context. Namely, they consider j a discrete and faithful representation of the fundamental group of a geometrically finite hyperbolic n manifold into $\text{Isom}(\mathbb{H}^n)$. On the other side, they also assume that ρ takes values in the isometry group of \mathbb{H}^n , but their method would work when replacing \mathbb{H}^n by a complete simply connected Riemannian manifold of curvature ≤ -1 , since it relies on an equivariant Kirzbraun–Valentine theorem than mainly requires the target space to be $CAT(-1)$.

Given a representation $\rho : \pi_1(S) \rightarrow \text{Isom}(M, g_M)$, it is natural to ask whether ρ can be dominated by a Fuchsian representation and, if so, what the domain of Fuchsian representations dominating ρ looks like. The first question was answered by Deroín and the author in [6], when the Riemannian manifold (M, g_M) has sectional curvature ≤ -1 . This applies for instance when M is the symmetric space of a simple Lie group of real rank 1 (with the suitable renormalization of the metric), and in particular for representations in $\text{PSL}(2, \mathbb{R})$.

Theorem (Deroín–Tholozan). *Let S be a surface of negative Euler characteristic, (M, g_M) a smooth complete simply connected Riemannian manifold, and ρ a representation of $\pi_1(S)$ into $\text{Isom}(M)$. If (M, g_M) has sectional curvature bounded above by -1 , then either ρ is Fuchsian in restriction to some stable totally geodesic 2-plane of curvature -1 embedded in M , or there exists a Fuchsian representation that strictly dominates ρ .*

Remark 0.4. This theorem was obtained independently and with other methods by Guéritaud–Kassel–Wolff [11], in the case where ρ takes values in $\text{PSL}(2, \mathbb{R})$. Note that, by a simple volume argument, a Fuchsian representation cannot be strictly dominated. Thus the theorem is optimal.

0.3. Topology of $\text{Adm}(S)$. Consider a representation $j : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{R})$ and a representation $\rho : \pi_1(S) \rightarrow \text{Isom}(M, g_M)$. One easily checks that $\text{Lip}(j, \rho)$ only depends on the class $[j]$ of j under conjugation by $\text{PSL}(2, \mathbb{R})$ and on the class $[\rho]$ of ρ under conjugation by $\text{Isom}(M, g_M)$. In particular, the function $j \mapsto \text{Lip}(j, \rho)$ induces a function on the space of classes of representations of Euler class $\chi(S)$ modulo conjugation, which is well-known to be isomorphic to the *Teichmüller space* of S , denoted $\mathcal{T}(S)$ (see 1.2.2). We will denote by $D(\rho)$ the domain of $\mathcal{T}(S)$ consisting of classes of representations of Euler class $\chi(S)$ strictly dominating a given representation ρ .

To prove their theorem, Deroin and the author consider in [6] a certain map $\Psi_\rho : \mathcal{T}(S) \rightarrow \mathcal{T}(S)$ and show that its image lies in $D(\rho)$. The map Ψ_ρ is constructed this way: given a point X in $\mathcal{T}(S)$ one can associate to the pair (X, ρ) a holomorphic quadratic differential $\Phi(X, \rho)$ on X , obtained as the Hopf differential of an equivariant harmonic map (see section 1.2.1). Then, by a theorem of Sampson–Hitchin–Wolf (see section 1.2.2) there is a unique Fuchsian representation j up to conjugacy such that $\Phi(X, j) = \Phi(X, \rho)$. We then set $\Psi_\rho(X) = [j]$. More details are given in section 2.1. Here, we prove that this map is a homeomorphism from $\mathcal{T}(S)$ to $D(\rho)$. We thus obtain the following:

Theorem 1. *Let S be a closed oriented surface of negative Euler characteristic, (M, g_M) a smooth complete simply connected Riemannian manifold of sectional curvature bounded above by -1 , and ρ a representation of $\pi_1(S)$ into $\text{Isom}(M, g_M)$ that is not Fuchsian in restriction to some stable totally geodesic 2-plane of curvature -1 . Then $D(\rho)$ is homeomorphic to an open ball of dimension $-3\chi(S)$.*

Assume now that (M, g_M) is the symmetric space of some simple Lie group G of real rank 1, with g_M normalized so that its sectional curvature is everywhere ≤ -1 . Denote by $\text{Hom}(\pi_1(S), G)$ the space of representations of $\pi_1(S)$ into G , and by $\text{Rep}(\pi_1(S), G)$ the algebraic quotient (in the sense of the geometric invariant theory) of $\text{Hom}(\pi_1(S), G)$ under the action of G by conjugation. Then the space of pairs (j, ρ) with j Fuchsian and $\rho \in \text{Hom}(\pi_1(S), G)$ strictly dominated by j fibers over a domain of $\mathcal{T}(S) \times \text{Rep}(S, G)$ (see 2.1.1). We denote this domain

$$\text{Dom}(S, (M, g_M)).$$

Let $\text{Rep}_{nf}(\pi_1(S), (M, g_M))$ be the space of classes of representations of $\pi_1(S)$ into G that are not Fuchsian in restriction to some stable totally geodesic copy of \mathbb{H}^2 of curvature -1 . The map Ψ_ρ only depends on the class of ρ in $\text{Rep}_{nf}(S, (M, g_M))$. One can thus define the map

$$\begin{aligned} \Psi : \mathcal{T}(S) \times \text{Rep}_{nf}(S, (M, g_M)) &\rightarrow \mathcal{T}(S) \times \text{Rep}_{nf}(S, (M, g_M)) \\ (X, [\rho]) &\mapsto (\Psi_\rho(X), [\rho]). \end{aligned}$$

By [6] and theorem 1, this map is a bijection from $\mathcal{T}(S) \times \text{Rep}_{nf}(S, (M, g_M))$ to $\text{Dom}(S, (M, g_M))$. Using standard arguments, one can prove that Ψ is continuous. In section 2.4, we will prove that its inverse is also continuous. We thus get the following:

Theorem 2. *Let S be a closed oriented surface of negative Euler characteristic and G a simple Lie group of real rank 1, with symmetric space (M, g_M) , where g_M is normalized to have sectional curvature ≤ -1 . Then $\text{Dom}(S, (M, g_M))$ is homeomorphic to $\mathcal{T}(S) \times \text{Rep}_{nf}(S, (M, g_M))$.*

Remark 0.5. This theorem should be true more generally for (M, g_M) any smooth complete simply connected Riemannian manifold of curvature ≤ -1 , provided that the map Ψ_ρ depends continuously on ρ for the natural topology on the space of representations of $\pi_1(S)$ to

$\text{Isom}(M, g_M)$. This, however, raises some technical questions about the topology of the character variety and the continuity of the energy functional (see 1.3). We thus chose to restrict ourselves to the situation where (M, g_M) is a symmetric space of rank 1, which includes the most interesting case of (\mathbb{H}^2, g_P) .

As a consequence of theorem 2, the connected components of $\text{Dom}(S, (M, g_M))$ are in 1-1 correspondance with the connected components of $\text{Rep}_{nf}(S, (M, g_M))$. In the case where M is the Poincaré half-plane, we have, by Kassel's theorem:

$$\text{Dom}(S, (\mathbb{H}^2, g_P)) = \text{Adm}(S).$$

Denoting by $\text{Rep}_k(S, \text{PSL}(2, \mathbb{R}))$ the algebraic quotient (under the action of $\text{PSL}(2, \mathbb{R})$) of the space of representations of $\pi_1(S)$ into $\text{PSL}(2, \mathbb{R})$ with Euler class k , we obtain:

Corollary 3. *Let S be a closed oriented surface of negative Euler characteristic. Then the space $\text{Adm}(S)$ is homeomorphic to*

$$\mathcal{T}(S) \times \bigsqcup_{\chi(S) < k < -\chi(S)} \text{Rep}_k(S, \text{PSL}(2, \mathbb{R})).$$

In particular, it has $4g - 5$ connected components, classified by the Euler class of the non-Fuchsian representation in each pair.

0.4. Further applications. Note that the map Ψ depends non trivially on the choice of a normalization of the metric on M . Fix a metric g_0 on M of sectional curvature ≤ -1 and a constant $\alpha \geq 1$. Then the metric $\frac{1}{\alpha}g_0$ still has sectional curvature ≤ -1 . To mark the dependence of the function Lip on the metric on the target, we denote by $\text{Lip}_g(j, \rho)$ the minimal Lipschitz constant of a (j, ρ) -equivariant map from (\mathbb{H}^2, g_P) to (M, g) . Then we clearly have $\text{Lip}_{\frac{1}{\alpha}g_0} = \frac{1}{\alpha}\text{Lip}_{g_0}$. Hence, if we apply theorem 2 to $(M, \frac{1}{\alpha}g_0)$, we obtain a description of the space of pairs $([j], [\rho]) \in \mathcal{T}(S) \times \text{Rep}(\pi_1(S), G)$ such that $\text{Lip}_{g_0}(j, \rho) < \alpha$.

Here is an application of this remark. Recall that Thurston's *asymmetric distance* on $\mathcal{T}(S)$ is the function on $\mathcal{T}(S) \times \mathcal{T}(S)$ defined by

$$d_{Th}([j], [j']) = \log \text{Lip}_{g_P}([j], [j'])$$

for j and j' two representations of Euler class $\chi(S)$. The function d_{Th} is continuous, positive whenever $[j]$ and $[j']$ are distinct, and satisfies the triangular inequality, but it is not symmetric.

Fix a point $[j_0]$ in $\mathcal{T}(S)$ and a constant $C > 0$. Then, using the convexity of length functions on $\mathcal{T}(S)$ [29], one can see that the domain

$$\{[j] \in \mathcal{T}(S) | d_{Th}([j_0], [j]) < C\}$$

is an open convex domain of $\mathcal{T}(S)$ for the Weil–Petersson metric. In particular, it is homeomorphic to a ball of dimension $-3\chi(S)$. Since the distance is asymmetric, it is not clear whether the same holds for

$$\{[j] \in \mathcal{T}(S) | d_{Th}([j], [j_0]) < C\}.$$

However, we have $d_{Th}([j], [j']) < C$ if and only if the pair $([j], [j'])$ is in

$$\text{Dom}(S, (\mathbb{H}^2, e^{-C}g_P)).$$

Now, the curvature of the metric $e^{-C}g_P$ is constant equal to $-e^{2C} < -1$, so we can apply theorems 1 and 2, and we obtain

Corollary 4. *Let S be a closed oriented surface of negative Euler characteristic. For any constant $C > 0$, we have:*

- *for any point $[j_0]$ in $\mathcal{T}(S)$, the domain $\{[j] \in \mathcal{T}(S) | d_{Th}([j], [j_0]) < C\}$ is homeomorphic to an open ball of dimension $-3\chi(S)$,*
- *the domain $\{([j], [j']) \in \mathcal{T}(S)^2 | d_{Th}([j], [j']) < C\}$ is homeomorphic to an open ball of dimension $-6\chi(S)$.*

We do not know whether this was known before, or could be deduced from convexity results on length functions.

Structure of the article and strategy of the proof. The article is organized as follows. In the next section we recall some fundamental results about harmonic maps from a surface. In particular, we recall the Corlette–Labourie theorem of existence of equivariant harmonic maps and the Sampson–Hitchin–Wolf parametrization of the Teichmüller space by means of quadratic differentials. In the second section, we start by using those theorems to construct the map Ψ_ρ studied in [6], and we prove that Ψ_ρ and Ψ are homeomorphisms. To do so we make explicit the inverse of the map Ψ_ρ . It will appear that reverse images of a point X in $\mathcal{T}(S)$ by Ψ_ρ are exactly critical points of a certain functional $\mathbf{F}_{X, [\rho]}$. We will prove that when X is in $D(\rho)$, the functional $\mathbf{F}_{X, [\rho]}$ is proper and admits a unique critical point which is a global minimum. Hence the map Ψ_ρ is bijective. What's more, the functionals $\mathbf{F}_{X, [\rho]}$ vary continuously with $(X, [\rho])$, and so does their unique minimum. This will prove the continuity of Ψ_ρ^{-1} and Ψ^{-1} .

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1. REPRESENTATIONS OF SURFACE GROUPS, HARMONIC MAPS, AND TEICHMÜLLER SPACE

In this section, we introduce briefly the tools from the theory of harmonic maps that we will need later. We refer to [5] for a more thorough survey.

1.1. Existence theorems. Recall that the *energy density* of a non-negative Riemann tensor h on a Riemannian manifold (X, g) is the function on X defined as

$$e_g(h) = \frac{1}{2} \text{Tr}(A(h)),$$

where $A(h)$ is the unique field of endomorphisms of the tangent bundle such that for all $x \in X$ and all $u, v \in T_x X$,

$$h_x(u, v) = g_x(u, A(h)_x v).$$

If f is a map between two Riemannian manifolds (X, g_X) and (Y, g_Y) then its energy density is the energy density on X of the pull-back metric:

$$e_{g_X}(f) = e_{g_X}(f^* g_Y).$$

This energy density can be integrated against the volume form Vol_{g_X} associated to g_X , giving the *total energy* of f :

$$E_{g_X}(f) = \int_X e_{g_X}(f) \text{Vol}_{g_X}.$$

The map f is called *harmonic* if it is, in some sense, a critical point of the total energy. For instance, harmonic maps from \mathbb{R} to a Riemannian manifold are geodesics, and harmonic maps from a Riemannian manifold to \mathbb{R} are harmonic functions. In general, a map f is harmonic if it satisfies a certain partial differential equation that can be expressed as the vanishing of the *tension field*. We give and use the precise definition of harmonicity in [6]. Here we will be satisfied with existence results and some fundamental properties of those maps.

The first existence result is due to Eells and Sampson.

Theorem (Eells–Sampson, [7]). *Let $f : (X, g_X) \rightarrow (Y, g_Y)$ be a continuous map between two closed Riemannian manifolds. Assume (Y, g_Y) has non-positive sectional curvature. Then there exists a harmonic map $f' : (X, g_X) \rightarrow (Y, g_Y)$ homotopic to f , which minimizes the energy among all maps homotopic to f . Moreover, if the sectional curvature of Y is negative, then this map is unique, unless the image of f' is included in a geodesic of Y .*

Eells and Sampson’s paper contains a thorough study of the analytic aspects of harmonicity, which allows to extend their existence result in several cases. The one we will be interested in is an equivariant version. Consider (X, g_X) a closed Riemannian manifold, (\tilde{X}, \tilde{g}_X) its universal cover, (Y, g_Y) another Riemannian manifold, and ρ a representation of $\pi_1(X)$ into $\text{Isom}(Y, g_Y)$. A map $f : \tilde{X} \rightarrow Y$ is called $(\pi_1(X), \rho)$ -equivariant if for all $x \in \tilde{X}$ and all $\gamma \in \pi_1(X)$, we have

$$f(\gamma \cdot x) = \rho(\gamma) \cdot f(x).$$

Given such a map, the Riemann tensor f^*g_Y on \tilde{X} is invariant under the action of $\pi_1(X)$ and thus induces to a Riemann tensor on X . We define the energy density and the total energy of f as the energy density and the total energy of this Riemann tensor.

The equivariant version of Eells–Sampson’s theorem is due to Corlette [4] in the specific case where Y is a symmetric space of non-compact type, and to Labourie [18] in the more general case of a complete simply connected Riemannian manifold of non-positive curvature. We state it in the particular case where the sectional curvature of (Y, g_Y) is bounded above by a negative constant. This implies that (Y, g_Y) is *Gromov hyperbolic*. Recall that, for such a space, there is a good notion of *boundary at infinity*, such that every isometry of Y extends to a homeomorphism of the boundary.

Theorem (Corlette, Labourie). *Let (X, g_X) be a closed Riemannian manifold, \tilde{X} its universal cover, and (Y, g_Y) a complete simply connected Riemannian manifold of sectional curvature bounded away from 0. Let ρ be a representation of $\pi_1(X)$ into $\text{Isom}(Y)$. Assume that ρ either has no fixed point in $\partial_\infty Y$ or fixes a geodesic in Y . Then there exists a harmonic map from (\tilde{X}, \tilde{g}_X) to (Y, g_Y) that is $(\pi_1(X), \rho)$ -equivariant. This map minimizes the energy among all such equivariant maps. If ρ does not fix a geodesic, then this map is unique.*

1.2. Harmonic maps from a surface, Hopf differential, and Teichmüller space.

From now on we will restrict to harmonic maps from a Riemann surface. In that case, the energy of a map only depends on the conformal class of the Riemannian metric on the base. For the same reason, harmonicity is invariant under conformal changes.

1.2.1. Hopf differential. Let S be an oriented surface equipped with a Riemannian metric g . Let (M, g_M) be a Riemannian manifold, and $f : S \rightarrow M$ a smooth map. The conformal class of g induces a complex structure on S . The Riemann tensor f^*g_M can thus be uniquely decomposed into a $(1, 1)$ part, a $(2, 0)$ part and a $(0, 2)$ part. One can check that the $(1, 1)$ part is $e_g(f)g$, and we thus have

$$f^*g_M = e_g(f)g + \Phi_f + \bar{\Phi}_f,$$

where Φ_f is a *quadratic differential* (i.e. a section of the bundle $K_{S,[g]}^2$), called the *Hopf differential* of f .

The following proposition is classical.

Proposition 1.1. *If the map f is harmonic, then its Hopf differential is holomorphic. The converse is true if M is also a surface.*

1.2.2. *The Teichmüller space.* Consider S a closed oriented surface of negative Euler characteristic. Recall that the Teichmüller space of S , denoted $\mathcal{T}(S)$, is the space of complex structures on S compatible with the orientation, where two complex structures are identified if there is a biholomorphism between them isotopic to the identity.

By Poincaré–Koebe’s uniformization theorem, any complex structure on S admits a unique conformal Riemannian metric of constant curvature -1 . Therefore, $\mathcal{T}(S)$ can also be seen as the space of hyperbolic metrics on S up to isotopy. Lastly, any hyperbolic metric g on S induces a Fuchsian representation $j : \pi_1(S) \rightarrow \mathrm{PSL}(2, \mathbb{R})$, such that (S, g) identifies isometrically with $j(\pi_1(S)) \backslash \mathbb{H}^2$ (j is called a *holonomy representation* of the metric g). The representation j is unique up to conjugation, has Euler class $\chi(S)$, and $\mathcal{T}(S)$ thus identifies canonically with the space of representations from $\pi_1(S)$ to $\mathrm{PSL}(2, \mathbb{R})$ of Euler class $\chi(S)$ modulo conjugation.

It is well known that the Teichmüller space is a manifold diffeomorphic to a ball of dimension $-3\chi(S)$ and that it carries a complex structure. Consider two points X_1 and X_2 in $\mathcal{T}(S)$, corresponding to two hyperbolic metrics g_1 and g_2 on S . Then, by Eells–Sampson’s theorem, there is a unique harmonic map $f_{g_1, g_2} : (S, g_1) \rightarrow (S, g_2)$ homotopic to the identity map. Schoen–Yau’s theorem [22] implies that this map is a diffeomorphism. (Note that Schoen–Yau’s theorem is actually more general. This particular case was also proved independently by Sampson [21].)

Of course, the map f_{g_1, g_2} depends on the choice of g_1 and g_2 up to isotopy. Actually, fixing g_1 or g_2 , one can choose the other metric so that the identity map itself is harmonic (by replacing g_2 by $f_{g_1, g_2}^* g_2$ or g_1 by $f_{g_1, g_2}^* g_1$). On the other hand, the total energy of f_{g_1, g_2} is invariant under changing one of the metrics by isotopy, and thus gives a well defined functional

$$\mathbf{E} : \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathbb{R}.$$

Besides, the Hopf differential of f_{g_1, g_2} is invariant under isotopic changes of g_2 , and if h is a diffeomorphism of S , one has

$$\Phi_{f_{h^*g_1, g_2}} = h^* \Phi_{f_{g_1, g_2}}.$$

The Hopf differential thus induces a well defined map

$$\Phi : \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathrm{QDT}(S),$$

where $\mathrm{QDT}(S)$ denotes the complex bundle of holomorphic quadratic differentials on $\mathcal{T}(S)$, that is, $\mathrm{QD}_X \mathcal{T}(S)$ is the space of holomorphic quadratic differentials on X .

Sampson [21] proved that this map is an injective immersion. In particular, if one fixes a point $X_0 \in \mathcal{T}(S)$, the derivative at X_0 of the map $\Phi(X_0, \cdot)$ provides an identification of the tangent space $T_{X_0} \mathcal{T}(S)$ with $\mathrm{QD}_{X_0} \mathcal{T}(S)$. (It is known since the work of Teichmüller that the bundle $\mathrm{QDT}(S)$ identifies with the tangent space to $\mathcal{T}(S)$.)

Lastly, Wolf [28] proved that the map Φ is a global homeomorphism. This was obtained independently by Hitchin [12], as the first construction of a section to the *Hitchin fibration*.

Theorem (Sampson, Hitchin, Wolf). *The map $\Phi : \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathrm{QDT}(S)$ is a homeomorphism.*

1.2.3. *The Weil–Petersson metric on $\mathcal{T}(S)$.* Consider $X \in \mathcal{T}(S)$, and choose g a hyperbolic metric representing X . Then g naturally induces a Hermitian metric on the line bundle K_X^2 . In a local conformal coordinate z , write

$$g = \alpha dz d\bar{z}$$

for some positive function α . Then a section Φ of K_X^2 is locally of the form φdz^2 , and the Hermitian metric on K_X^2 induced by g is given by

$$\|\Phi\|_g^2 = \frac{|\varphi|^2}{\alpha^2}.$$

(One can check that this function is independent of the choice of a local conformal coordinate.)

This Hermitian metric allows to define a Hermitian norm on the complex vector space $\text{QD}_X \mathcal{T}(S)$ of holomorphic quadratic differentials on X , given by

$$\|\Phi\|_{WP}^2 = \int_S \|\Phi\|_g^2 \text{Vol}_g.$$

Since $\text{QD}_X \mathcal{T}(S)$ naturally identifies with $\text{T}_X \mathcal{T}(S)$, this defines a Riemannian metric on $\mathcal{T}(S)$, called the *Weil–Petersson metric*. The Weil–Petersson metric is a Kähler metric [1]. Many interesting properties of the Weil–Petersson metric have been proved by Ahlfors [1], Wolpert [29], and many others. Here we will be satisfied with this simple definition.

1.3. **Equivariant harmonic maps and functionals on $\mathcal{T}(S)$.** Consider now a complete simply connected Riemannian manifold (M, g_M) of sectional curvature bounded above by -1 . Let S be a closed oriented surface of negative Euler characteristic, and denote by

$$\text{Hom}_{ss}(\pi_1(S), \text{Isom}(M))$$

the space of representations of $\pi_1(S)$ into $\text{Isom}(M)$ that either don't fix a point in $\partial_\infty M$ or fix a geodesic. This condition ensures that the action of $\text{Isom}(M)$ on $\text{Hom}_{ss}(\pi_1(S), \text{Isom}(M))$ by conjugation has closed orbits. We denote the quotient under this action by

$$\text{Rep}_{ss}(\pi_1(S), \text{Isom}(M)).$$

When M is the symmetric space of some simple Lie group G of rank 1, the following properties are equivalent:

- (i) the representation ρ either does not fix a point in $\partial_\infty M$ or fixes a geodesic,
- (ii) the orbit of ρ under conjugation by G is closed in the space of representations,
- (iii) ρ is *semi-simple*, i.e. for any linear representation $\tau : G \rightarrow \text{GL}(V)$, any subspace of V invariant by $\tau \circ \rho$ admits an invariant complement.

Since the action of G on $\text{Hom}(\pi_1(S), G)$ is algebraic, Mumford's *geometric invariant theory* [19] implies that the fiber over a point of the algebraic quotient $\text{Rep}(\pi_1(S), G)$ contains a unique closed orbit. Therefore, $\text{Rep}_{ss}(\pi_1(S), G)$ can be identified with $\text{Rep}(\pi_1(S), G)$.

The theorem of Corlette and Labourie allows to extend the maps \mathbf{E} and Φ to $\mathcal{T}(S) \times \text{Rep}_{ss}(\pi_1(S), \text{Isom}(M))$. Given a point $X_0 \in \mathcal{T}(S)$, represented by a hyperbolic metric g_0 , and a point $[\rho]$ in $\text{Rep}_{ss}(\pi_1(S), \text{Isom}(M))$, one can consider $f_{g_0, \rho}$ a $(\pi_1(S), \rho)$ -equivariant harmonic map from (\tilde{S}, \tilde{g}_0) to (M, g_M) . The Riemann tensor $f_{g_0, \rho}^* g_M$ on \tilde{S} is $\pi_1(S)$ -invariant, hence the energy density and the Hopf differential of $f_{g_0, \rho}$ are also $\pi_1(S)$ -invariant. They thus descend on S . We denote them $e_{g_0}(f_{g_0, \rho})$ and $\Phi_{f_{g_0, \rho}}$.

If ρ does not fix a point in $\partial_\infty M$, then $f_{g_0, \rho}$ is unique. If ρ fixes a geodesic in M , then the image of $f_{g_0, \rho}$ lies in this geodesic, and all such maps are the same up to a translation along

this geodesic. In any case, the energy density and the Hopf differential only depend on ρ . We eventually obtain two well-defined maps

$$\begin{aligned} \mathbf{E} : \mathcal{T}(S) \times \text{Rep}_{ss}(\pi_1(S), \text{Isom}(M)) &\rightarrow \mathbb{R}_+ \\ (X_0, [\rho]) &\mapsto \int_S e_{g_0}(f_{g_0, \rho}) \text{Vol}_0 \end{aligned}$$

and

$$\begin{aligned} \Phi : \mathcal{T}(S) \times \text{Rep}_{ss}(\pi_1(S), \text{Isom}(M)) &\rightarrow QDT(S) \\ (X_0, [\rho]) &\mapsto \Phi_{f_{g_0, \rho}}. \end{aligned}$$

Remark 1.2. If $M = \mathbb{H}^2$, we have $\text{Isom}^+(M) \simeq \text{PSL}(2, \mathbb{R})$. Consider two points $X, X_1 \in \mathcal{T}(S)$, and let j_1 be a holonomy representation of a hyperbolic metric g_1 corresponding to X_1 . Then a harmonic map from X to X_1 isotopic to the identity lifts to a $(\pi_1(S), j_1)$ -equivariant harmonic map from \tilde{X} to \mathbb{H}^2 . We thus have

$$\mathbf{E}(X, [j_1]) = \mathbf{E}(X, X_1)$$

and

$$\Phi(X, [j_1]) = \Phi(X, X_1).$$

In other words, the new definition of \mathbf{E} and Φ extends the one given in the previous paragraph, via the identification of $\mathcal{T}(S)$ with the component of minimal Euler class in $\text{Rep}(\pi_1(S), \text{PSL}(2, \mathbb{R}))$.

Proposition 1.3. *The function $\mathbf{E}(X, \rho)$ and the map $\Phi(X, \rho)$ are continuous in X . If M is a rank 1 symmetric space, these are also continuous in ρ .*

Proof. This is a classical consequence of the ellipticity of the equation defining harmonic maps. If (j_n, ρ_n) converges to (j, ρ) , the derivatives of a (j_n, ρ_n) -equivariant harmonic f_n can be uniformly controlled by its total energy. The hypothesis that ρ_n and ρ are semisimple gives a control on $f_n(p)$ for some base point p (see [4]). One then deduces that the sequence f_n converges in C^1 topology to a (j, ρ) -equivariant harmonic map f . The proposition easily follows. \square

We will use the following important results.

Proposition 1.4 (see [27]). *Let (M, g_M) be a complete simply connected Riemannian manifold of sectional curvature bounded away by -1 , and ρ a representation in $\text{Hom}_{ss}(\pi_1(S), \text{Isom}(M))$. Then the functional*

$$\begin{aligned} \mathbf{E}(\cdot, [\rho]) : \mathcal{T}(S) &\rightarrow \mathbb{R}_+ \\ X &\mapsto \mathbf{E}(X, [\rho]) \end{aligned}$$

is C^1 , and its differential is given by

$$\mathbf{Grad}_{WP} \mathbf{E}(\cdot, [\rho])(X_0) = -\Phi(X_0, [\rho])$$

(where \mathbf{Grad}_{WP} denotes the gradient with respect to the Weil–Petersson metric).

Theorem 1.5 (Tromba, [26]). *Consider a point Y in $\mathcal{T}(S)$. Then the function*

$$\begin{aligned} \mathbf{E}(\cdot, Y) : \mathcal{T}(S) &\rightarrow \mathbb{R}_+ \\ X &\mapsto \mathbf{E}(X, Y) \end{aligned}$$

is proper.

2. THE HOMEOMORPHISM FROM $\mathcal{T}(S)$ TO $D(\rho)$

We are now in possession of all the tools required to define the map Ψ_ρ introduced in [6], and to prove that it is a homeomorphism from $\mathcal{T}(S)$ to $D(\rho)$.

2.1. Construction of the map Ψ_ρ . Let S be a closed oriented surface of negative Euler characteristic and ρ a representation of $\pi_1(S)$ into the isometry group of a complete simply connected Riemannian manifold (M, g_M) of sectional curvature bounded above by -1 . Assume ρ either has no fixed point in $\partial_\infty M$ or fixes a geodesic in M . Let X_1 be a point in $\mathcal{T}(S)$. Then $\Phi(X_1, [\rho])$ is a holomorphic quadratic differential on X_1 , and the theorem of Sampson–Hitchin–Wolf asserts that there is a unique point X_2 in $\mathcal{T}(S)$ such that $\Phi(X_1, X_2) = \Phi(X_1, [\rho])$. Setting

$$\Psi_\rho(X_1) = X_2,$$

we obtain a well-defined map

$$\Psi_\rho : \mathcal{T}(S) \rightarrow \mathcal{T}(S).$$

This map only depends on the class of ρ under conjugation by $\text{Isom}(M)$. We can thus define a map

$$\begin{aligned} \Psi : \mathcal{T}(S) \times \text{Rep}_{ss}(\pi_1(S), \text{Isom}(M, g_M)) &\rightarrow \mathcal{T}(S) \times \text{Rep}_{ss}(\pi_1(S), \text{Isom}(M, g_M)) \\ (X, [\rho]) &\mapsto (\Psi_\rho(X), [\rho]). \end{aligned}$$

In [6], the following is proved:

Theorem (Deroin–Tholozan). *Either ρ preserves a totally geodesic 2-plane of curvature -1 in restriction to which it is Fuchsian, or the image of the map Ψ_ρ lies in $D(\rho)$. In particular, $D(\rho)$ is non empty.*

2.1.1. Dealing with representations fixing a point at infinity. We only defined Ψ_ρ for a representation $\rho \in \text{Hom}_{ss}(\pi_1(S), \text{Isom}(M))$. However, if ρ fixes a point in $\partial_\infty M$, then ρ preserves a Buseman function on M , and one can find a representation ρ' in $\text{PSL}(2, \mathbb{R})$ fixing a geodesic in \mathbb{H}^2 , such that $l(\rho(\gamma)) = l(\rho'(\gamma))$ for any $\gamma \in \pi_1(S)$ (see [6], section 2.1). We hence have:

$$D(\rho) = D(\rho'),$$

since, by Guéritaud–Kassel’s theorem (see section 0.2.1), j strictly dominates ρ if and only if the length spectrum of j strictly dominates the length spectrum of ρ .

In the case where M is the symmetric space of a simple Lie group G of rank 1, a consequence of Guéritaud–Kassel’s theorem is that the domain $D(\rho)$ is constant on each fiber of the projection $\pi : \text{Hom}(\pi_1(S), G) \rightarrow \text{Rep}(\pi_1(S), G)$. Indeed, for any $\gamma \in \pi_1(S)$, the function $\rho \mapsto l(\rho(\gamma))$ is a continuous function invariant under the action of G , and it is therefore constant on the fibers of the algebraic quotient.

The space of pairs (j, ρ) with j strictly dominating ρ thus fibers over a domain $\text{Dom}(S, (M, g_M))$ of $\mathcal{T}(S) \times \text{Rep}(\pi_1(S), G)$, and since $\text{Rep}(\pi_1(S), G)$ identifies with $\text{Rep}_{ss}(\pi_1(S), G)$, the map Ψ will completely describe $\text{Dom}(S, (M, g_M))$.

2.2. Surjectivity of the map Ψ_ρ . We first prove the following

Proposition 2.1. *The map $\Psi_\rho : \mathcal{T}(S) \rightarrow D(\rho)$ is surjective.*

Proof. Fix $X_0 \in \mathcal{T}(S)$ and $\rho \in \text{Hom}_{ss}(\pi_1(S), \text{Isom}(M, g_M))$. Let us introduce the functional

$$\begin{aligned} \mathbf{F}_{X_0, [\rho]} : \mathcal{T}(S) &\rightarrow \mathbb{R} \\ X &\mapsto \mathbf{E}(X, X_0) - \mathbf{E}(X, [\rho]). \end{aligned}$$

By theorem 1.4, the map $\mathbf{F}_{X_0, [\rho]}$ is C^1 , and its gradient with respect to the Weil–Petersson metric is given by

$$\text{Grad}_X(\mathbf{F}_{X_0, [\rho]}) = \Phi(X, X_0) - \Phi(X, [\rho]).$$

Hence, X_1 is a critical point of $\mathbf{F}_{X_0, [\rho]}$ if and only if $\Phi(X_1, X_0) = \Phi(X_1, [\rho])$, which means that

$$\Psi_\rho(X_1) = X_0.$$

Proving that X_0 is in the image of Ψ_ρ is thus equivalent to proving that the map $\mathbf{F}_{X_0, [\rho]}$ admits a critical point. This will be a consequence of the following lemma:

Lemma 2.2. *For $X_0 \in \mathcal{T}(S)$ and $\rho \in \text{Hom}_{ss}(\pi_1(S), \text{Isom}(M))$, we have the following inequality:*

$$\mathbf{F}_{X_0, [\rho]} \geq (1 - \text{Lip}(X_0, \rho)) \mathbf{E}(\cdot, X_0).$$

Proof. Let g_0 be a hyperbolic metric representing X_0 . For any $\varepsilon > 0$, let $f : (\tilde{S}, \tilde{g}_0) \rightarrow (M, g_M)$ be a $(\pi_1(S), \rho)$ -equivariant Lipschitz map with Lipschitz constant $\text{Lip}(X_0, \rho) + \varepsilon$. Let X be a point in $\mathcal{T}(S)$, represented by some hyperbolic metric g , and let h be a diffeomorphism of S isotopic to the identity, such that $h : (S, g) \rightarrow (S, g_0)$ is harmonic. We thus have $E_g(h^*g_0) = \mathbf{E}(X, X_0)$. Let \tilde{h} be the lift of h to \tilde{S} . Then the map $f \circ \tilde{h}$ is a $(\pi_1(S), \rho)$ -equivariant map from \tilde{S} to M , and we thus have

$$\mathbf{E}(X, [\rho]) \leq E_g(f \circ \tilde{h}).$$

Since f is $(\text{Lip}(X_0, \rho) + \varepsilon)$ -lipschitz, we have

$$\mathbf{E}(X, \rho) \leq E_g(f \circ \tilde{h}) \leq (\text{Lip}(X_0, \rho) + \varepsilon) E_g(h^*g_0) = (\text{Lip}(X_0, \rho) + \varepsilon) \mathbf{E}(X, X_0),$$

From which we get

$$\mathbf{F}_{X_0, [\rho]}(X) = \mathbf{E}(X, X_0) - \mathbf{E}(X, [\rho]) \geq (1 - \text{Lip}(X_0, [\rho]) - \varepsilon) \mathbf{E}(X, X_0).$$

This being true for any $\varepsilon > 0$, we obtain the desired inequality. \square

Now, by 1.5, the map $X \rightarrow \mathbf{E}(X, X_0)$ is proper. Therefore, if X_0 is in $D(\rho)$, the function $\mathbf{F}_{X_0, [\rho]}$ is also proper. Hence $\mathbf{F}_{X_0, [\rho]}$ admits a minimum. This minimum is a critical point, and thus a pre-image of X_0 by the map Ψ_ρ . We obtain that Ψ_ρ is surjective. \square

Remark 2.3. We proved that if X_0 is in $D(\rho)$, the functional $\mathbf{F}_{X_0, [\rho]}$ is proper. Conversely, if $\mathbf{F}_{X_0, [\rho]}$ is proper, then it admits a critical point. Hence X_0 is in the image of Ψ_ρ , which implies that X_0 lies in $D(\rho)$ by [6]. We thus obtain the following corollary that might be interesting:

Corollary 2.4. *The map $\mathbf{F}_{X_0, [\rho]}$ is proper if and only if X_0 lies in $D(\rho)$.*

2.3. Injectivity of the map Ψ_ρ . To prove injectivity, we need to prove that when $X_0 \in D(\rho)$, the critical point of $\mathbf{F}_{X_0, [\rho]}$ is unique. To do so, we prove that any critical point of $\mathbf{F}_{X_0, [\rho]}$ is a strict minimum of $\mathbf{F}_{X_0, [\rho]}$.

Let X_1 be a critical point of $\mathbf{F}_{X_0, [\rho]}$, and X_2 another point in $\mathcal{T}(S)$. Choose a hyperbolic metric g_1 on S representing X_1 , and let g_0 be the hyperbolic metric representing X_0 such that $\text{Id} : (S, g_1) \rightarrow (S, g_0)$ is harmonic, and g_2 the hyperbolic metric representing X_2 such that $\text{Id} : (S, g_2) \rightarrow (S, g_0)$ is harmonic. Let $f : (\tilde{S}, \tilde{g}_1) \rightarrow (M, g_M)$ be a $(\pi_1(S), \rho)$ -equivariant harmonic map. We have the following decompositions:

$$\begin{aligned} g_0 &= e_{g_1}(g_0)g_1 + \Phi + \bar{\Phi}, \\ f^*g_M &= e_{g_1}(f)g_1 + \Phi + \bar{\Phi}, \\ g_2 &= e_{g_1}(g_2)g_1 + \Psi + \bar{\Psi}, \end{aligned}$$

where Φ and Ψ are quadratic differentials on S , with Φ holomorphic with respect to the complex structure induced by $[g_1]$. Note that the same Φ appears in the decomposition of g_0 and f^*g_M because X_1 is a critical point of $F_{X_0, \rho}$, and thus $\Phi(X_1, X_0) = \Phi(X_1, [\rho])$.

Lemma 2.5. *We have the following identity:*

$$(1) \quad E_{g_2}(g_0) - E_{g_2}(f^*g_M) = \int_S \frac{1}{\sqrt{1 - \frac{4\|\Psi\|_{g_1}^2}{e_{g_1}(g_2)^2}}} (e_{g_1}(g_0) - e_{g_1}(f^*g_M)) \text{Vol}_{g_1}.$$

Proof of lemma 2.5. This is a rather basic computation that we will carry out in local coordinates. Let $z = x + iy$ be a local complex coordinate with respect to which g_1 is conformal. We denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on \mathbb{C}

Any Riemann tensor on \mathbb{C} can be written under the form $\langle \cdot, G \cdot \rangle$, where G is a field of symmetric endomorphisms of \mathbb{R}^2 depending on the coordinates (x, y) . We will represent such an endomorphism by its matrix in the canonical frame $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. In local coordinates, we can thus write

$$\begin{aligned} g_0 &= \langle \cdot, G_0 \cdot \rangle, \\ g_1 &= \langle \cdot, G_1 \cdot \rangle, \\ g_2 &= \langle \cdot, G_2 \cdot \rangle, \\ f^*g_M &= \langle \cdot, G_f \cdot \rangle. \end{aligned}$$

Now, since g_1 is conformal with respect to the coordinate z , we have $g_1 = \alpha \langle \cdot, \cdot \rangle$ for some positive function α , and we can write

$$\begin{aligned} \Phi &= \varphi dz^2, \\ \Psi &= \psi dz^2, \end{aligned}$$

for some complex valued functions φ and ψ . (Since Φ is holomorphic, φ must be holomorphic, but we won't need it for our computation.)

We can now express $G_0, G_1, G_2, G_f, \text{Vol}_{g_1}$ and Vol_{g_2} in terms of $\alpha, e_{g_1}(g_0), e_{g_1}(g_2), e_{g_1}(f), \varphi$ and ψ . We easily check that

$$\begin{aligned} G_1 &= \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \\ G_0 &= \begin{pmatrix} \alpha e_{g_1}(g_0) + 2\text{Re}(\varphi) & -2\text{Im}(\varphi) \\ -2\text{Im}(\varphi) & \alpha e_{g_1}(g_0) - 2\text{Re}(\varphi) \end{pmatrix}, \\ G_f &= \begin{pmatrix} \alpha e_{g_1}(g_f) + 2\text{Re}(\varphi) & -2\text{Im}(\varphi) \\ -2\text{Im}(\varphi) & \alpha e_{g_1}(g_f) - 2\text{Re}(\varphi) \end{pmatrix}, \\ G_2 &= \begin{pmatrix} \alpha e_{g_1}(g_2) + 2\text{Re}(\psi) & -2\text{Im}(\psi) \\ -2\text{Im}(\psi) & \alpha e_{g_1}(g_2) - 2\text{Re}(\psi) \end{pmatrix}, \\ \text{Vol}_{g_1} &= \alpha dz d\bar{z}, \end{aligned}$$

$$\text{Vol}_{g_2} = \sqrt{\det G_2} dz d\bar{z} = \sqrt{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2} dz d\bar{z}.$$

We now want to express $e_{g_2}(g_0)$ and $e_{g_2}(f^*g_M)$. To do so, note that we can write

$$\begin{aligned} g_0(\cdot, \cdot) &= \langle \cdot, G_0 \cdot \rangle \\ &= \langle \cdot, G_2(G_2^{-1}G_0) \cdot \rangle \\ &= g_2(\cdot, G_2^{-1}G_0 \cdot). \end{aligned}$$

By definition of the energy density, we thus obtain that

$$\begin{aligned} e_{g_2}(g_0) &= \frac{1}{2} \text{Tr}(G_2^{-1}G_0) \\ &= \frac{1}{2} \text{Tr} \left(\frac{1}{\det G_2} \begin{pmatrix} \alpha e_{g_1}(g_2) - 2\text{Re}(\psi) & 2\text{Im}(\psi) \\ 2\text{Im}(\varphi) & \alpha e_{g_1}(g_2) + 2\text{Re}(\psi) \end{pmatrix} \begin{pmatrix} \alpha e_{g_1}(g_0) + 2\text{Re}(\varphi) & -2\text{Im}(\varphi) \\ -2\text{Im}(\varphi) & \alpha e_{g_1}(g_0) - 2\text{Re}(\varphi) \end{pmatrix} \right) \\ &= \frac{1}{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2} (\alpha^2 e_{g_1}(g_2) e_{g_1}(g_0) - (\varphi \bar{\psi} + \bar{\varphi} \psi)). \end{aligned}$$

Simimilarly, we get that

$$e_{g_2}(f^*g_M) = \frac{1}{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2} (\alpha^2 e_{g_1}(g_2) e_{g_1}(f) - (\varphi \bar{\psi} + \bar{\varphi} \psi)).$$

When computing the difference, the terms $\varphi \bar{\psi} + \bar{\varphi} \psi$ simplify. (Here we use the fact that f^*g_M and g_0 have the same $(2,0)$ -part.) We eventually obtain

$$\begin{aligned} (e_{g_2}(g_0) - e_{g_2}(f^*g_M)) \text{Vol}_{g_2} &= \frac{\alpha^2 e_{g_1}(g_2)(e_{g_2}(g_0) - e_{g_2}(f^*g_M))}{\sqrt{\alpha^2 e_{g_1}(g_2)^2 - 4|\psi|^2}} dz d\bar{z} \\ &= \frac{e_{g_2}(g_0) - e_{g_2}(f^*g_M)}{\sqrt{1 - 4 \frac{\|\Psi\|_{g_1}^2}{e_{g_1}(g_2)^2}}} \text{Vol}_{g_1}. \end{aligned}$$

Now, the parameters of the last expression are well defined functions on S , and the identity is true in any local chart. It is thus true everywhere on S , and integrating, we obtain lemma 2.5. \square

From (1), we obtain that

$$E_{g_2}(g_0) - E_{g_2}(f^*g_M) \geq \int_S (e_{g_1}(g_0) - e_{g_1}(f^*g_M)) \text{Vol}_{g_1} = E_{g_1}(g_0) - E_{g_1}(f^*g_M) = \mathbf{F}_{X_0, \rho}(X_1),$$

with equality if and only if $\|\Psi\|_{g_1} \equiv 0$, that is, if g_1 is conformal to g_2 .

On the other side, we have $E_{g_2}(g_0) = \mathbf{E}(X_2, X_0)$ (since, by hypothesis, the identity map from (S, g_2) to (S, g_0) is harmonic) and $E_{g_2}(f^*g_M) \geq \mathbf{E}(X_2, [\rho])$, from which we deduce that

$$E_{g_2}(g_0) - E_{g_2}(f^*g_M) \leq \mathbf{F}_{X_0, [\rho]}(X_2).$$

Combining the two inequalities, we obtain that

$$\mathbf{F}_{X_0, [\rho]}(X_2) \geq \mathbf{F}_{X_0, [\rho]}(X_1),$$

with equality if and only if $X_1 = X_2$.

Now, if X_1 and X_2 are two critical points of $\mathbf{F}_{X_0, [\rho]}$, then by symmetry we must have $\mathbf{F}_{X_0, [\rho]}(X_2) = \mathbf{F}_{X_0, [\rho]}(X_1)$, and therefore $X_2 = X_1$. The functional $\mathbf{F}_{X_0, [\rho]}$ admits a unique critical point, and X_0 admits a unique reverse image by Ψ_ρ . Thus Ψ_ρ is injective.

2.4. bi-continuity of Ψ_ρ and Ψ . Recall that one has, by definition,

$$(X, \Psi_\rho(X)) = \Phi^{-1}(X, \Phi(X, [\rho])),$$

where Φ^{-1} denotes the inverse of the map $\Phi : \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \text{QDT}(S)$, which is a homeomorphism by Sampson–Hitchin–Wolf’s theorem. Therefore, by proposition 1.3, the maps Ψ_ρ and Ψ are continuous.

Let us now assume that (M, g_M) is the symmetric space of some simple Lie group G of rank 1, and prove that Ψ^{-1} is continuous. It will be clear that, if one fixes ρ and allows only X to vary, then the same proof would give the continuity of Ψ_ρ^{-1} in the more general setting where (M, g_M) is a complete simply connected Riemannian space of sectional curvature ≤ -1 .

We saw that $\Psi^{-1}(X, [\rho])$ is the unique critical point of a proper function $\mathbf{F}_{X,\rho}$ on $\mathcal{T}(S)$ which depends continuously on X and ρ . Hence, the continuity of Ψ^{-1} is a consequence of the fact that the functions $\mathbf{F}_{X,\rho}$ are in some sense *locally uniformly proper*.

Definition 2.6. Let X and Y be two metric spaces, and $(F_y)_{y \in Y}$ a family of continuous functions from X to \mathbb{R} depending continuously on y for the compact-open topology. We say that the family $(F_y)_{y \in Y}$ is *uniformly proper* if for any $C \in \mathbb{R}$, there exists a compact $K \subset X$ such that for all $y \in Y$ and all $x \in X \setminus K$, $F_y(x) > C$.

We say that the family $(F_y)_{y \in Y}$ is *locally uniformly proper* if for any $y_0 \in Y$, there is a neighbourhood U of y_0 such that the sub-family $(F_y)_{y \in U}$ is uniformly proper.

Proposition 2.7. *Let X and Y be two metric spaces, and $(F_y)_{y \in Y}$ a locally uniformly proper family of continuous functions from X to \mathbb{R} depending continuously on y (for the compact-open topology). Assume that each F_y achieves its minimum at a unique point $x_m(y) \in X$. Then the function*

$$y \mapsto x_m(y)$$

is continuous.

Proof. Let us denote by $m(y) = F_y(x_m(y))$ the minimum value of F_y . Fix $y_0 \in Y$. Let U be a neighbourhood of y_0 and K a compact of X such that for all $y \in U$ and all $x \in X \setminus K$, we have

$$F_y(x) > m(y_0) + 1.$$

For $\varepsilon > 0$, define

$$V_\varepsilon = \{x \in X \mid F_{y_0}(x) < m(y_0) + \varepsilon\}.$$

Since F_{y_0} is proper and achieves its minimum at a single point $x_m(y_0)$, the family $(V_\varepsilon)_{\varepsilon > 0}$ forms a basis of neighbourhoods of $x_m(y_0)$. Let U_ε be a neighbourhood of y_0 included in U , such that for all $y \in U_\varepsilon$ and all $x \in K$,

$$|F_y(x) - F_{y_0}(x)| \leq \frac{\varepsilon}{2}.$$

(U_ε exists because the map $y \mapsto F_y$ is continuous for the compact-open topology.) Since $x_m(y_0)$ is obviously in K , we have for all $y \in U_\varepsilon$,

$$F_y(x_m(y_0)) \leq m(y_0) + \frac{\varepsilon}{2},$$

hence the minimum value $m(y)$ of F_y is smaller than $m(y_0) + \frac{\varepsilon}{2}$. In particular, for $\varepsilon < 2$, this minimum is achieved in K (since outside K , we have $F_y \geq m(y_0) + 1$). We thus have $x_m(y) \in K$, from which we deduce

$$F_{y_0}(x_m(y)) \leq F_y(x_m(y)) + \frac{\varepsilon}{2} = m(y) + \frac{\varepsilon}{2} \leq m(y_0) + \varepsilon.$$

We have thus proved that for all $y \in U_\varepsilon$, $x_m(y) \in V_\varepsilon$. Since $(V_\varepsilon)_{\varepsilon>0}$ is a basis of neighbourhoods of $x_m(y_0)$, this proves that $y \mapsto x_m(y)$ is continuous at y_0 . \square

To prove the continuity of Ψ^{-1} , we can apply proposition 2.7 to the family $\mathbf{F}_{X,[\rho]}$ of functions on $\mathcal{T}(S)$ depending on the parameter $(X, [\rho]) \in \text{Dom}(S, (M, g_M))$. The continuity of $(X, [\rho]) \mapsto \mathbf{F}_{X,[\rho]}$ comes from proposition 1.3. The only thing we need to check is thus that the family

$$(\mathbf{F}_{X,[\rho]})_{(X,[\rho]) \in \text{Dom}(S, (M, g_M))}$$

is locally uniformly proper. This will be a consequence of the following lemma:

Lemma 2.8. *The function Lip is upper-semicontinuous on $\mathcal{T}(S) \times \text{Hom}(\pi_1(S), G)$.*

Remark 2.9. The continuity of Lip is proved by Guéritaud–Kassel [10] in a more general setting (see remark 0.3). In their paper they assume that ρ takes values in $\text{Isom}(\mathbb{H}^n)$, but their method probably generalizes to representations into a rank 1 Lie group. However, we will be satisfied here with upper-semicontinuity, which we will prove by a standard argument consisting in seeing a representation as the monodromy of a flat connexion on a G -bundle.

proof of lemma 2.8. For clarity, we will prove separately that $\text{Lip}(X, \rho)$ is continuous in X and upper-semicontinuous in ρ . However, it will be clear that, using both arguments simultaneously, one would prove upper-semicontinuity in (X, ρ) .

First, let us fix $\rho \in \text{Hom}(\pi_1(S), \text{Isom}(M))$ and prove that the map $X \rightarrow \text{Lip}(X, \rho)$ is continuous. We will use the continuity of Thurston's asymmetric distance [25]. Let X be a point in $\mathcal{T}(S)$, represented by a hyperbolic metric g_X . Fix $\varepsilon > 0$, and let U_ε be a neighbourhood of X such that for all $Y \in U_\varepsilon$, we have $\text{Lip}(X, Y) \leq 1 + \varepsilon$ and $\text{Lip}(Y, X) \leq 1 + \varepsilon$ (the existence of such a neighbourhood is equivalent to the continuity of Thurston's asymmetric distance). Fix a point $Y \in U_\varepsilon$ and a hyperbolic metric g_Y representing Y . Since Y is in U_ε , there exists a map $h : (S, g_Y) \rightarrow (S, g_X)$ homotopic to the identity and $(1 + 2\varepsilon)$ -Lipschitz. Denote by \tilde{h} its lift to \tilde{S} . Finally, consider $f_X : (\tilde{S}, \tilde{g}_X) \rightarrow M$ a $(\pi_1(S), \rho)$ -equivariant Lipschitz map with Lipschitz constant $\text{Lip}(X, \rho) + \varepsilon$. Then

$$f_X \circ \tilde{h} : (\tilde{S}, \tilde{g}_Y) \rightarrow M$$

is a $(\pi_1(S), \rho)$ -equivariant Lipschitz map with Lipschitz constant $(1 + 2\varepsilon)(\text{Lip}(X, \rho) + \varepsilon)$, and we thus have

$$\text{Lip}(Y, \rho) \leq \text{Lip}(X, \rho) + \varepsilon(1 + 2\varepsilon + \text{Lip}(X, \rho)).$$

Exchanging X and Y , we get similarly

$$\text{Lip}(X, \rho) \leq \text{Lip}(Y, \rho) + \varepsilon(1 + 2\varepsilon + \text{Lip}(Y, \rho)).$$

We eventually obtain:

$$\text{Lip}(X, \rho) - C\varepsilon \leq \text{Lip}(Y, \rho) \leq \text{Lip}(X, \rho) + C\varepsilon,$$

for a constant C depending only on X . This gives the continuity of $Y \mapsto \text{Lip}(Y, \rho)$.

Let us now prove the upper-semicontinuity in ρ . The space $\text{Hom}(\pi_1(S), G)$ is locally arcwise connected, and we thus need to prove that, given a point $X \in \mathcal{T}(S)$ and a continuous family ρ_t of representations defined for t in neighbourhood of 0, we have

$$(2) \quad \limsup_{t \rightarrow 0} \text{Lip}(X, \rho_t) \leq \text{Lip}(X, \rho_0).$$

Each representation ρ_t is canonically the holonomy of a flat connexion on the twisted bundle

$$E_t = \tilde{S} \times M / \pi_1(S),$$

Where the action of $\pi_1(S)$ is given by

$$\gamma \cdot (x, y) = (\gamma \cdot x, \rho_t(\gamma) \cdot y).$$

Since ρ_t is a continuous deformation of ρ_0 , it is classical that all the bundles E_t are topologically identical as G -bundles, and ρ_t can thus be seen as the monodromy of a connection ∇_t on a fixed G -bundle E with fiber M . The connection ∇_t is well-defined up to an automorphism of the bundle E , and one can choose ∇_t so that it varies continuously with t .

Let us now recall that, for each t , there is a natural correspondance between sections of the flat bundle (E, ∇_t) and $(\pi_1(S), \rho_t)$ -equivariant maps from \tilde{S} to M . Let g_X be a hyperbolic metric on S representing X . Fix $\varepsilon > 0$ and let $f_0 : (\tilde{S}, \tilde{g}_X) \rightarrow M$ be a $(\pi_1(S), \rho_0)$ -equivariant Lipschitz map with Lipschitz constant $\text{Lip}(X, \rho_0) + \varepsilon$. This map induces a section s of the flat bundle (E, ∇_0) , and the same section s , seen as a section of (E, ∇_t) , induces a $(\pi_1(S), \rho_t)$ -equivariant map $f_t : (\tilde{S}, \tilde{g}_X) \rightarrow M$. Now, the fact ∇_t varies continuously with t implies that f_t is Lipschitz with Lipschitz constant

$$\text{Lip}(X, \rho_0) + \varepsilon + \alpha(t),$$

for some continuous function $\alpha(t)$ such that $\alpha(0) = 0$. From this, one easily deduces that

$$\limsup_{t \rightarrow 0} \text{Lip}(X, \rho_t) \leq \text{Lip}(X, \rho_0) + \varepsilon.$$

This being true for any $\varepsilon > 0$, we obtain (2), which proves the upper-semicontinuity of $\rho \mapsto \text{Lip}(X, \rho)$. \square

Corollary 2.10. *The family $(\mathbf{F}_{X, [\rho]})_{(X, [\rho]) \in \text{Dom}(S, (M, g_M))}$ is locally uniformly proper on $\mathcal{T}(S)$.*

Proof. Let $(X_0, [\rho_0])$ be a point in $\text{Dom}(S, (M, g_M))$. We thus have $\text{Lip}(X_0, \rho_0) < 1$. By lemma 2.8, Lip is upper-semicontinuous. Hence there exists a neighbourhood U of $(X_0, [\rho_0])$ such that for all $(X, [\rho]) \in U$, we have $\text{Lip}(X, \rho) \leq \frac{1 + \text{Lip}(X_0, \rho_0)}{2}$. By lemma 2.2, we thus have

$$\mathbf{F}_{X, [\rho]}(Y) \geq \left(\frac{1 - \text{Lip}(X_0, \rho_0)}{2} \right) \mathbf{E}(Y, X)$$

for all $(X, [\rho]) \in U$ and all $Y \in \mathcal{T}(S)$. Since the function $Y \mapsto \mathbf{E}(Y, X)$ is proper by theorem 1.5, we obtain that the family $(\mathbf{F}_{X, [\rho]})$ is uniformly proper on U . Hence it is locally uniformly proper. \square

Since the family $(\mathbf{F}_{X, [\rho]})_{(X, [\rho]) \in \text{Dom}(S, (M, g_M))}$ is locally uniformly proper, by proposition 2.7, the unique minimum of $\mathbf{F}_{X, [\rho]}$ varies continuously with $(X, [\rho])$. Since this minimum is precisely $\Psi^{-1}(X, [\rho])$, we proved that Ψ^{-1} is continuous. Fixing ρ and allowing only X to vary, one would obtain the continuity of Ψ_ρ^{-1} in the more general setting of a representation ρ in the isometry group of a complete simply connected Riemannian manifold of sectional curvature ≤ -1 . This finishes the proof of theorems 1 and 2.

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